Eigen Values & Eigen Vectors:

If A is square matrix, then a vector X is said to be Eigen vector of the matrix A. if $\exists$ a number λ s.t.

$$\boxed{AX = \lambda X}$$

If A is a square matrix of order n then X is column matrix of order n.

The number λ is known as Eigen values or characteristic roots or latent roots.

$$AX = \lambda X.$$
$$AX = \lambda I X$$
$$AX - \lambda I X = 0$$
$$[A - \lambda I] X = 0$$

if A is square matrix then char. poly of matrix A is $|A - \lambda I|$

& characteristic eq^n is $|A - \lambda I| = 0$.

if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

then $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$

is characteristic poly. of A.

& ~~i.e~~ $|A - \lambda I| = 0$. is char. eq^n.

The roots of Char. eqn gives eigen values.
or Char. roots.

Note :- (i) If A is a square matrix of order 2 then

$|A-\lambda I|=0$ is

$$\lambda^2 - s_1\lambda + |A| = 0$$

where s_1 = sum of the principal diagonal elements.

i.e $[a_{11}+a_{22}]$

(ii) If A is a square matrix of order 3 then

$|A-\lambda I|=0$ is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0.$$

where s_1 = sum of principal diagonal elts.

i.e $[a_{11}+a_{22}+a_{33}]$.

& s_2 = sum of minors of the principal diagonal elements.

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Properties of Eigen Values :-

(1) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$.

sum of Eigen values = Trace of matrix.

(2) $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n$.

$|A|$ = product of eigen values.

(3). if λ $(\lambda \neq 0)$ is eigen value of A then $\frac{1}{\lambda}$ is eigen value of A^{-1}. (if exist).

(4) $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen values of A, then. A^m has eigen values, $\lambda_1^m, \lambda_2^m, \cdots, \lambda_n^m$

(m-+ve integer)

⑤ A & A^T have same eigen values.

⑥ $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of matrix A then KA has $K\lambda_1, K\lambda_2 \dots$ eigen values.
(K = non zero scalar)

⑦ λ is eigen value of A, then $A \pm KI$ has $\lambda \pm K$ eigen value.

Steps to find Eigen vectors.

① Properties

① Eigen vectors corresponding to distinct eigen values always linearly independent.

② X_1, X_2 are othogonal if A is symmetric
(x_1, x_2 = eigenvectors).

Steps : if Eigen Values are $\lambda = \lambda_1, \lambda_2, \lambda_3 \dots$

Ⓐ Non-repeated eigen values for un-symmetric & symmetric matrix.

① if eigen values are non repeated then find eigen vectors corresponding to each eigen values separately by following method.

Consider eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of a matrix

$[A - \lambda I]X = 0.$

- substitute value $\lambda = \lambda_1$, in a matrix $[A - \lambda I]X = 0$

This matrix gives three equations & by cramer rule find values of x, y, z gives X_1.

- Similarly X_2 & X_3 corresponding to $\lambda = \lambda_2$ & $\lambda = \lambda_3$.

B] Repeated eigen values for un symmetric matrix:

- If eigen values are $\lambda = \lambda_1, \lambda_2, \lambda_3$ & two values are repeated i.e $\lambda_2 = \lambda_3$
- First find eigen vector X_1 for λ_1 by above method.
- To find eigen vectors for $\lambda_2 = \lambda_3$.
  Find first rank of $[A - \lambda I]$ after substituting λ_2.
- If rank $= r$ & no. of unknowns $= n$.
  then $(n-r)$ no. of linearly independent Eigen vectors are possible.
- If only one eigen vector is possible then find it by above method given in (A).
- If two L.I vectors are possiple.
  There is only one eqn from last matrix.
  put $z=0$ & find x & y ~~eigen~~
  given vector X_2
  - Again in the same eqn put $y=0$ & x & z gives Eigen vector X_3.

C] Repeated eigen values for symmetric matrix

- if $\lambda_2 = \lambda_3$.
- Find X_1 for λ_1.
- for $\lambda_2 = \lambda_3$.
- find rank of $[A - \lambda I]$ for λ_2
- $n-r$ L.I vectors.
- If only one vector then same method.
- if two vectors then.
  then
  Find first eigen vector X_2 for $\lambda_2 = \lambda_3$ by (A)
  & To find third X_3, consider

$$X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

From eq^n $x_1 x_3^T = 0$ & $x_2 x_3^T = 0$ find values of l, m, n it gives x_3. (41)

Example:- Find Eigen values & Eigen vectors of the matrix. $A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$

$\Longrightarrow$ Given matrix is,

$$A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$$

The characteristic eq^n is, $|A - \lambda I| = 0$

$$\therefore [A - \lambda I] = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{bmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{vmatrix}.$$

it is of order 2×2

$$\therefore |A - \lambda I| = \lambda^2 - s_1 \lambda + |A| \quad (s_1 = \text{Trace of } A = 14 + (-1)).$$

$$= \lambda^2 - (13)\lambda + 36$$

$$= \lambda^2 - 13\lambda + 36.$$

which is quadratic equation,

$$\therefore (\lambda - 4)(\lambda - 9) = 0$$

$$\therefore \lambda - 4 = 0 \text{ \& } \lambda - 9 = 0$$

$$\therefore \lambda = 4, 9.$$

Hence Eigen values are $\lambda_1 = 4$ & $\lambda_2 = 9$.

{ Verificat^n $\lambda_1 + \lambda_2 = a_{11} + a_{22}$ | & $\lambda_1 \cdot \lambda_2 = |A|$

$4 + 9 = 14 - 1$ | $4 \times 9 = 36$

$13 = 13$ | $36 = 36$. }

Step-II : Eigen vectors.

To find eigen vectors $X = \begin{bmatrix} x \\ y \end{bmatrix}$

corresponding to eigen values $\lambda_1 = 4$, $\lambda_2 = 9$.

consider $[A - \lambda I]X = \begin{bmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$. — ①

For $\lambda_1 = 4$

$$[A - 4I]X = \begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{matrix} 10x - 10y = 0 \\ 5x - 5y = 0 \end{matrix}\right\} \Rightarrow x = y$$

(choose $x = 1$ to get smallest +ve int-)

$\therefore$ For $\lambda_1 = 4$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now for $\lambda_2 = 9$,

put $\lambda = 9$ in eqn ①.

$$\therefore [A - 9I]X = \begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

$$\Rightarrow \left.\begin{matrix} 5x - 10y = 0 \\ \& \ 5x - 10y = 0 \end{matrix}\right\}$$

$$\Rightarrow 5x - 10y = 0$$

$$\Rightarrow 5x = 10y$$

$$x = \frac{10}{5} y$$

$$\boxed{x = 2y}$$

choose $y = 1 \Rightarrow x = 2$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore$ For $\lambda_2 = 9$, Eigen vector $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Hence Eigen vectors are

$$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\lambda_1 = 4} \text{ \& } X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\lambda_2 = 9}$$

Example: Find Eigen values ~~of~~ & corresponding Eigen vectors for matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Example: Find Eigen values & Eigen vectors for the matrix.

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$\Longrightarrow$ for Given matrix.

Characteristic eq^n is

$$|A - \lambda I| = 0$$

$$\therefore [A - \lambda I] = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -1-\lambda & 1 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{vmatrix}$$

$$= \lambda^3 - S_1\lambda^2 + S_2\lambda - |A|.$$

S_1 = Trace of A (Given).

S_2 = Σ minors of principal diagonal elt.

$$\therefore S_1 = -1 - 2 - 3 = -6$$

$$S_2 = \begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix}$$

$S_2 = 6+3+2 = 11.$ & $|A| = -1(6-0) - 1(0) + 2(0) = -6$

$\therefore |A - \lambda I| = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0.$

$$\begin{array}{c|cccc} -1 & 1 & 6 & +11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1 & \cancel{6}\ 5 & +6 & \boxed{0} \end{array}$$

$\therefore \lambda = 1$ is first root.

$(\lambda+1)(\lambda^2+5\lambda+6) = 0$

$\lambda = -1$ & $\lambda^2+5\lambda+6=0$

$\cancel{\lambda^2 + 3\lambda + 3\lambda - 6} = 0$

$\cancel{\lambda(\lambda+2)}.$

$\lambda = -1, \quad \lambda^2 + 2\lambda + 3\lambda + 6 = 0$

$\lambda(\lambda+2) + 3(\lambda+2) = 0$

$(\lambda+2)(\lambda+3) = 0$

$\therefore \lambda = -1, \lambda = -2$ & $\lambda = -3$

{Note if triangle is upper triangular then diagonal elements are eigen values itself}

$\therefore$ For eigen vectors

consider, $[A - \lambda I]X = 0$

$$\begin{bmatrix} -1-\lambda & 1 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{—— (1)}$$

put $\lambda = -1$.

$$\therefore [A+I] = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 0x + y + 2z = 0$ —— (A)

$0x - y + z = 0$ —— (B)

$0x + 0y - 2z = 0$ —— (C)

$\Rightarrow \boxed{z=0}$.

or solve by cramer's rule.

put $z=0$ in eq^n (B).

$\therefore -y+0=0$

$\Rightarrow \boxed{y=0}$

but we can't say about x.

$\therefore$ x is parameter, say k.

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \quad \text{put } k=1$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\lambda=-1}.$$

Now for $\lambda_2=-2$, put $\lambda=-2$ in (1).

$$\therefore [A-\lambda I]X = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$\Rightarrow$ $x+y+2z=0$, $0x+0y+z=0$, $0x+0y-z=0$

~~$z=0$~~

$\therefore \boxed{z=0}$, but can't say about x & y.

$\therefore$ From (1), $x=-y$.

put $x=k \Rightarrow y=-k$.

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} \quad \text{put } k=1.$$

$$\therefore X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_{\lambda=-2}.$$

$S_n = 6 + 2 + 2$

For $\lambda_3 = -3$, put $\lambda = -3$ in (1).

$$\therefore [A - \lambda I]X = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + y + 2z = 0 \quad \text{— (i)}$$

$$y + z = 0$$

$$0x + 0y + 0z = 0.$$

$$y = -z$$

$\therefore$ put $y = k$.

$\therefore$ $z = -k$.

From (i)

$$2x + k - 2k = 0$$

$$2x - k = 0$$

$$2x = +k \Rightarrow x = k/2.$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k/2 \\ k \\ -k \end{bmatrix} \quad k = 2 \text{ to get smallest +ve int.}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \lambda = -3.$$

$\therefore$ Eigen vectors are, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Example: $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$, find eigen values & eigenvectors of A.

Non-repeated Eigen Values for symmetric matrix:- (17)

Q.1. Find the Eigen value & Eigen vectors for the following matrix: $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\Rightarrow$ Given matrix is,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic eq^n of given matrix A is,

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

$\therefore$ characteristic eq^n is,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$= \lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0.$$

s_1 = Trace of A.
$= 3 + 5 + 3$
$= 11$

s_2 = minor of principal diagonal elts.
$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$
$$= (15-1) + (9-1) + (15-1)$$
$$= 14 + 8 + 14 = 36.$$

$$|A| = 3(15-1) + 1(-3+1) + 1(1-5)$$
$$= 3(14) + 1(-2) + 1(-4)$$
$$= 42 - 2 - 4$$
$$= 36.$$

$$\therefore |A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_1\lambda - |A| = 0$$
$$= \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

Solve eqn by synthetic division.

$$\begin{array}{c|cccc} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\therefore \lambda = 2$ is first root.

$$\& \ \lambda^2 - 9\lambda + 18 = 0.$$
$$\lambda^2 - 3\lambda - 6\lambda + 18 = 0.$$
$$\lambda(\lambda - 3) - 6(\lambda - 3) = 0$$
$$\therefore \lambda = 3, 6.$$

$$\therefore \lambda_1 = 2, \lambda_2 = 3 \ \& \ \lambda_3 = 6.$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Repeated Eigen values

Q. Find Eigen values & Eigen vectors for the following matrix

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

$\Longrightarrow$

The characteristic eq^n is

$|A - \lambda I| = 0$

$$\therefore A - \lambda I = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix} \quad \text{——} \; (1)$$

$$\therefore |A - \lambda I| = \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix}$$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0.$$

S_1 = Trace of A.

$= -9 + 3 + 7$

$= 1$

$$S_2 = \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix}$$

$= (21 - 32) + (-63 + 64) + (-27 + 32)$

$= -11 + 1 + 5$

$= -5$

$|A| = -9(21 - 32) - 4(-56 + 64) + 4(-64 + 48)$

$= -9(-11) - 4(8) + 4(-16)$

$|A| = 3$

$\therefore |A - \lambda I| = \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -5 & -3 \\ & & -1 & 2 & 3 \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

$(\lambda = -1)$ & $\lambda^2 - 2\lambda - 3 = 0$

$\lambda^2 + \lambda - 3\lambda - 3 = 0$

$\lambda(\lambda+1) - 3(\lambda+1) = 0$

$(\lambda+1)(\lambda-3) = 0$

$\therefore \lambda_1 = -1,\ \lambda_2 = -1$ & $\lambda = 3$

$\therefore$ -1 is repeated.

$\therefore$ First find eigen vector for $\lambda = 3$ by same method

put $\lambda = 3$ in eqn 1.

$\therefore [A - \lambda I]X = 0$

$\therefore [A - 3I]X = 0$

$$\therefore \begin{bmatrix} -9-3 & 4 & 4 \\ -8 & 3-3 & 4 \\ -10 & 8 & 7-\cancel{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -10 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

& $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\therefore$ eqn becomes.

$-12x + 4y + 4z = 0$

$-8x + 0y + 4z = 0$

$-16x + 8y + 4z = 0$.

$$\frac{x}{\begin{vmatrix} 0 & 4 \\ 8 & 4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -8 & 4 \\ -16 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -8 & 0 \\ -16 & 8 \end{vmatrix}} = k.$$

$$\frac{x}{0-32} = \frac{-y}{-32-(-64)} = \frac{z}{-64-0} = k.$$

$$\frac{x}{-32} = \frac{-y}{32} = \frac{z}{-64} = k.$$

$\therefore x = -32k$

$y = -32k$

$z = -64k$

if we take $k = -\frac{1}{32}$ (we get samallest +ve int).

$\therefore x = 1, y = 1, z = 2$

$\therefore$ For $\lambda = 3$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Now For $\lambda = -1$ i.e $\lambda_2 = \lambda_3 = -1$.

in eq^n ①.

$$[A - \lambda I][X] = \begin{bmatrix} -9-(-1) & 4 & 4 \\ -8 & 3-(-1) & 4 \\ -16 & 8 & 7-(-1) \end{bmatrix} X = 0$$

$$\begin{bmatrix} -9+1 & 4 & 4 \\ -8 & 3+1 & 4 \\ -16 & 8 & 7+1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operate.

$R_2 - R_1$, $R_3 - 2R_1$ for coeff matrix.

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\therefore$ Rank $= 1$ & $n = 3$.

$\therefore n - r = 3 - 1 = 2$ l.I vectors possible.

$\therefore -8x + 4y + 4z = 0$

put $z = 0$, $-8x + 4y = 0$

$\Rightarrow -2x + y = 0$

$\Rightarrow y = 2x$.

put $x = 1 \Rightarrow y = 2$ & $z = 0$..

$$\therefore X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Put $y = 0$ $-8x + 4z = 0$

$\Rightarrow -2x + z = 0$

$z = 2x$

put $x = 1 \Rightarrow z = 2$ & $y = 0$.

$$\therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Q.2. Find Eigen values & Eigen vectors for the following matrix :

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, \quad \lambda = -3, -3, 5.$$

$X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Examples on repeated Eigen values for symmetric Matrix.

Q.1. Find Eigen values & Eigen vectors for the following matrix A : $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\Rightarrow$

$|A - \lambda I| = 0$ is characteristic eq^n of A.

$$\therefore A - \lambda I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0.$$

$S_1 =$ Trace of A.

$= 3+3+3$

$= 9$

$S_2 = \Sigma$ minor of diagonal elts

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$= (9-1) + (9-1) + (9-1).$

$= 8+8+8$

$= 24$

& $|A| = 3(9-1) - 1(3+1) + 1(-1-3)$

$= 3(8) - 1(4) + 1(-4)$

$= 24-4-4$

$= 16.$

$\therefore |A - \lambda I| = \lambda^3 \, S_1\lambda^2 + S_2\lambda - 16$

$= \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$

To find λ,

1	1	-9	24	-16
		1	-8	16
	1	-8	16	$\boxed{0}$

$\therefore (\lambda-1)(\lambda^2 - 8\lambda + 16) = 0$

$\therefore (\lambda-1) = 0$ & $(\lambda^2 - 8\lambda + 16) = 0.$

$\boxed{\lambda_1 = 1}$, $\lambda^2 - 4\lambda - 4\lambda + 16 = 0$

$\lambda(\lambda-4) - 4(\lambda-4) = 0$

$(\lambda-4)(\lambda-4) = 0.$

$\therefore \lambda_1 = 1$ & $\lambda_2 = 4$, $\lambda_3 = 4$

To find eigen vectors $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to

Eigen values $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 4$.

$\therefore$ consider $[A - \lambda I]X = 0$.

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{— } (1)$$

Put $\lambda = 1$ For $\lambda_1 = 1$

$\therefore$ (1) becomes.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ eq^n becomes,

$$2x + y + z = 0$$
$$x + 2y - z = 0$$
$$x - y + 2z = 0$$

Solve by using cramers Rule.

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = k.$$

$$\frac{x}{(4-1)} = \frac{-y}{(2+1)} = \frac{z}{(-1-2)} = k.$$

$$\frac{x}{3} = \frac{-y}{3} = \frac{z}{-3} = k.$$

$\Rightarrow x = 3k, \; y = -3k \;\&\; z = -3k.$

put $k = 1/3$ or $-\frac{1}{3}$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}_{\lambda_1 = 1}$$

Now for $\lambda_2 = \lambda_3 = 4$ put $\lambda = 4$ in (1).

$$\therefore [A - 4I] = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

Here eigen values are repeated

$\therefore$ Find rank of $[A-\lambda I]$

Operate R_2+R_1, R_3+1.

$$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A-\lambda I) = 1.$

i.e $r=1$ & $n=3$

$\therefore n-r = 3-1 = 2$

$\therefore$ Two eigen vectors are possible.

& A is symmetric

$\therefore$ Eigen vectors are orthogonal.

$\therefore$ from above matrix

eqn is, $-x+y+z=0$

Put $z=0$

$\Rightarrow -x+y=0 \Rightarrow x=y$

put $x=k$ $\therefore y=k$, $k=1$

$$\therefore X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\lambda_2=4}.$$

Now as we know eigen vectors are orthogonal

$\therefore X_1 X_3^T = 0$ & $X_2 X_3^T = 0$ if $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

Hence, $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} [l\ m\ n] = 0$ & $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} [l\ m\ n]$.

$-l+m+n=0$ & $l+m+0n = 0 \Rightarrow l+m=0$.

Solving eqn simultaneous.

put $m = k$.

$\therefore$ from second eq^n.

$\boxed{l = -m} \Rightarrow \boxed{l = -k}$.

From ① first eq^n.

$-l + m + n = 0$

$-(-k) + k + n = 0$

$2k + n = 0$

$\Rightarrow \boxed{n = -2k}$.

$$\therefore X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -k \\ k \\ -2k \end{bmatrix} \text{ put } k = -1$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \lambda_3 = 4.$$

Q find Eigen values & independent Eigen vectors for the following matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

* Caley-Hamilton Theorem:

statement: Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic eq^n:

Example: verify caley Hamilton Theorem & hence find A^{-1} for the matrix.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ Also find A^4 :

⇒ As Caley-Hamilton Theorem states that Every square matrix satisfies its own eq^n.

∴ Characteristic eq^n for A is

$$|A - \lambda I| = 0.$$

$$\therefore [A - \lambda I] = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$$\therefore |A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & -\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

S_1 = Trace of A.
$= 2+2+2$
$= 6$

S_2 = Σ minors of diagonal elts.

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & +2 \end{vmatrix}$$

$= (4-1) + (4-1) + (4-1)$
$= 3 + (3) + (3)$
$= 3 + 3 + 3$
$= 9$

$|A| = 2(4-1) + 1(-2+1) + 1(1-2)$
$= 2(3) + 1(-1) + 1(-1)$
$= 6 - 1 - 1 = 6 - 2$
$= 4$

$\therefore |A - \lambda I| = 0$

$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Now verify Caley Hamilton th^m, put $\lambda = A$.

$\therefore$ we show that

$A^3 - 6A^2 + 9A - 4 = 0.$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & 21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 9A - 4I =$

$$\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & 21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^3 - 6A^2 + 9A - 4I = 0 \quad (1)$$

Hence Caley-Hamilton th^m verified.

To find A^{-1}, multiply eq^n 1 by A^{-1}.

$\therefore A^{-1}A^3 - 6A^{-1}A^2 + 9A^{-1}A - 4A^{-1}I = 0$

$\therefore A^2 - 6A + 9I - 4A^{-1} = 0$

$\therefore 4A^{-1} = A^2 - 6A + 9I$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

To find A^4, multiply eq^n 1 by A.

$\therefore AA^3 - 6AA^2 + 9A\cdot A - 4A\cdot I = 0.$

$A^4 - 6A^3 + 9A^2 - 4A = 0.$

$\therefore A^4 = 6A^3 - 9A^2 + 4A.$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 9\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}.$$

Q.2 Verify Cayley Hamilton th^m for $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

& hence find A^{-1}.